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Marvin Carlton Papenfuss
Iowa State University

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Marvin Carlton Papenfuss

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I. INTRODUCTION

Although Liouville [14] used successive approximations in his study of differential equations as early as 1840, the power of this method was not fully appreciated until tedious hand computation was replaced by electronic computation. Computers have also been a prime factor contributing to further theoretical research on the subject of successive approximations as applied to many different areas of mathematics, including differential equations.

Successive approximations for differential equations have been particularly useful in the study of initial value problems. A standard format is to write the differential equation as an equivalent integral equation, to choose an approximate solution in some fashion, and to generate a sequence of successive approximations for the integral equation. Existence and/or uniqueness of a solution is then obtained by using a fixed point theorem, such as the contraction principle. Finally, an approximate solution can then be obtained on the computer, since the existence is constructive in the sense that not only is existence and uniqueness established but also the solution is in fact the limit of the sequence of successive approximations.

On the other hand, the study of boundary value problems has not been so richly enhanced by the use of successive approximations. There is, however, some usage, for example, successive approximations were used to study the general n^{th} order linear boundary value problem in 1926 [9]. Analytic function theory was used to obtain sufficient conditions for convergence.

Consider the nonlinear two-point boundary value problem

$$y'' = h(t,y,y'); \quad y(0) = A, \quad y(T) = B, \tag{1.1}$$

where $T > 0$ is fixed. All methods of successive approximations for the solving of (1.1)[1] involve the defining of the k^{th} approximation in terms of one or more of the preceding approximations. There are, however, many ways to proceed toward this goal, most of which involve changing the differential equations into integral equations. These integral equations generally contain a Green's function kernel. Roughly speaking, a Green's function is a generalized solution to a linear homogeneous boundary value problem. Thus, the term "linear" enters into the study of successive approximations for the nonlinear problem (1.1). Consequently, different ways of linearizing (1.1) will give rise to different definitions of successive approximations. Perhaps the simplest linearization of (1.1) is to define the linearization to be identically zero on $[0, T]$. The differential equations solved by the successive approximations for $k = 1,2,\cdots$ take on the form

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[1]Herein, the equations will be represented by ( ) and the references will be represented by [ ].

$$y''_{k+1} = h(t,y_k,y'_k); \quad y_{k+1}(0) = A, \quad y_{k+1}(T) = B. \tag{1.2}$$

The equivalent integral equations are then

$$y_{k+1} = -\int_0^T G_0(t,s)\ h(s,\ y_k(s),y'_k(s))ds + \omega(t), \tag{1.3}$$

where $G_0(t,s)$ is the Green's function for the problem $y'' = 0$ and $\omega(t)$ is the linear function which satisfies

$$\omega(0) = A, \quad \omega(T) = B.$$

Therefore, finding the solution $y(t)$ of (1.1) is equivalent to solving the integral equation

$$y(t) = -\int_0^T G_0(t,s)\ h(s,y(s),y'(s))ds + \omega(t). \tag{1.4}$$

The use of the successive approximations defined by (1.3) to study the boundary value problem (1.1) was considered as early as 1893 by Picard, and a more recent study was done by Bailey, Shampine, and Waltman (BSW) [1]. Some of the assumptions made as well as some of the iteration schemes employed by BSW [1] are discussed below as motivation for the work that follows.

Several of the basic assumptions made by BSW are continuity of $h(t,y,y')$ on $[0, T] \times R^2$, where R is the set of real numbers, and Lipschitz conditions of various types for $h(t,y,y')$. One such Lipschitz condition is

$$K_1(y-x) \le h(t,y,y')-h(t,x,y') \le K_2(y-x) \quad \text{if} \quad y \ge x \tag{1.5a}$$

$$L_1(y'-x') \le h(t,y,y')-h(t,y,x') \le L_2(y'-x') \quad \text{if} \quad y' \ge x', \quad (1.5b)$$

where K_i, L_i, $i = 1, 2$, are either Lipschitz constants or continuous functions on the interval $[0, T]$. The Lipschitz constants (functions) are then assumed to be sufficiently small so that the iteration (1.3) is a contraction and, thus, uniform convergence is obtained for any initial function $y_1(t)$ contained in some class of functions.

Another iteration scheme discussed by BSW [1] is called the modified Picard iteration. This method is used for (1.1) with the assumptions that $h(t,y,y')$ is independent of y', say $h(t,y,y') = f(t,y)$, and that $A = B = 0$. The following theorem describes the procedure.

Theorem 1.1: If $f(t,y)$ is continuous on $[0, T] \times R$ and satisfies (1.5a) with $K_2 < \pi^2/T^2$, then (1.1) has a unique solution $y(t)$, and the iteration sequence $y_n(t)$ defined by

$$y_n'' + ky_n = ky_{n-1} + f(t,y_{n-1}); \quad y_n(0) = y_n(T) = 0,$$

where $k = \frac{1}{2}(K_1 + K_2)$, converges uniformly to $y(t)$.

The proof is again by means of the contraction mapping principle and, thus, $y_1(t)$ can be any member of some class of functions.

A third type of iteration scheme discussed by BSW [1] establishes monotonic convergence of the sequence of successive approximations. The method is called Newton's method, and again it is assumed that $h(t,y,y') = f(t,y)$ and that $A = B = 0$. The method involves the approximation of $f(t,y)$ by the first few terms of a Taylor series expansion. Although the convergence is monotonic, the price is also quite high, since $f(t,y)$ is assumed to be a concave or a convex function of y.

In the work that follows, monotonic convergence is established for a sequence of successive approximations for (1.1), where it is assumed throughout that $h(t,y,y')$ is linear in y'. The linearization of (1.1), more correctly called a quasi-linearization, which is employed was introduced by Lees [13]. Although he considered only functions $h(t,y,y')$ which were independent of y', an analogous quasi-linearization of the general problem (1.1) can be performed.

Suppose $h(t,y,y')$ has continuous first partial derivatives on $[0, T] \times R^2$, and let

$$u(t) = y(t) - \omega(t) \tag{1.6a}$$

$$A(t;u) = \int_0^1 h_{y'}(t,\xi u+\omega,\xi u'+\omega')d\xi \tag{1.6b}$$

$$B(t;u) = \int_0^1 h_y(t,\xi u+\omega,\xi u'+\omega')d\xi. \tag{1.6c}$$

Then it can be shown that (1.1) is equivalent to the boundary value problem

$$u'' = A(t;u)u'+B(t;u)u+h(t,\omega,\omega'); \quad u(0) = u(T) = 0. \tag{1.7}$$

A preview of how successive approximations will be used to solve (1.7) can now be given. Let $u_1(t)$ be qualitatively chosen from some known class of functions S. For all $u \in S$, make sufficient assumptions upon $A(t;u)$ and $B(t;u)$ to guarantee that the Green's functions $G_k(t,s)$ for the problems

$$u''_{k+1} = A(t;u_k)u'_{k+1} + B(t;u_k)u_{k+1}; \quad k = 1,2,3,\cdots, \tag{1.8}$$

exist, are unique, are continuous, and are nonnegative on $[0, T] \times [0, T]$. Next, define successive approximations $u_{k+1}(t)$ for (1.7) which are solutions to the boundary value problems

$$u''_{k+1} = A(t;u_k)u'_{k+1} + B(t;u_k)u_{k+1} + h(t,\omega,\omega');$$

$$u_{k+1}(0) = u_{k+1}(T) = 0. \tag{1.9a}$$

As integral equations, these successive approximations have the form

$$u_{k+1}(t) = -\int_0^T G_k(t,s)\, h(s,\omega(s),\omega'(s))\,ds. \tag{1.9b}$$

Finally, show convergence of $\{u_k(t)\}$, and thereby establish the existence and/or the uniqueness of a solution to (1.7). Therefore, by (1.6a), existence and/or uniqueness for (1.1) will have been established.

The best results of this thesis are included in Theorems

2.6, 2.14, 3.7, and 3.15, and Corollary 2.16. Corollaries 2.8, 2.9, and 3.12 and Note 3.13 involve particular boundary value problems, such as the no-damping case.

II. COMPARISON OF GREEN'S FUNCTIONS

Let L be the second order linear differential operator defined by

$$Ly = y'' - a(t)y', \tag{2.1}$$

where it is assumed that $a(t)$ is continuous on $[0, T]$. Let $b(t)$ be continuous on $[0, T]$, and denote by $G(t,s;b)$ the Green's function, if it exists, for the linear problem

$$Ly = b(t)y. \tag{2.2}$$

The notation $G_o(t,s)$ will be reserved for the case $a(t) = b(t) = 0$ on $[0, T]$.

Definition 2.1: Equation 2.2 is said to be disconjugate on $[0, T]$ if the only solution with two zeros on $[0, T]$ is the trivial solution $y \equiv 0$.

Disconjugacy criteria for the general n^{th} order linear homogeneous differential equation are quite abundant in the literature. In particular, disconjugacy results for (2.2) are very numerous, since second order equations arise frequently in applications. These criteria for (2.2) range from the fairly abstract, such as the result of Heimes [8] which shows that in a Banach space (2.2) is disconjugate on $[0, T]$ if $a(t) \equiv 0$ on $[0, T]$ and

$$\max_{0\le t\le T} \int_0^T |G_0(t,s)b(s)|ds \le 1,$$

to a result for the scalar case of (2.2) due to Fink [5] which states that (2.2) is disconjugate on $[0, T]$ if

$$-\int_0^T b^-(s)ds \le \frac{4}{T}\exp(-\frac{1}{2}\int_0^t |a(s)|ds), \tag{2.3}$$

where $b^-(t) = \min\{b(t),0\}$. Another very useful theorem on disconjugacy is the variational principle. Details can be found in [7]. A good summary of some of the more recent results on disconjugacy for the n^{th} order problem and, in particular, for (2.2), was given by Brink [3].

Let $u_j(t)$, $j = 1,2$, be functions which satisfy the initial conditions

$$u_j(\delta_{2j}T) = 0, \quad u_j'(\delta_{2j}T) = 1, \tag{2.4}$$

where δ_{ij} is the Kroneker delta.

Lemma 2.2: Suppose that Equation 2.2 is disconjugate on $[0, T]$. Then there exist unique solutions $u_j(t)$ which solve the problems (2.2) and (2.4), and which extend across the entire interval $[0, T]$. Furthermore, $u_1(t) > 0$ on $(0, T]$ and $u_2(t) < 0$ on $[0, T)$.

Proof: Existence, uniqueness, and extendibility are known since both problems are linear initial value problems.

By the construction of the functions u_j if the inequalities were not true the disconjugacy assumption would be violated.||

Theorem 2.3: If Equation 2.2 is disconjugate on $[0, T]$, then there exists a unique, continuous, nonnegative Green's function $G(t,s;b)$ on $[0, T] \times [0, T]$.

Proof: Proofs of existence, uniqueness, and continuity of $G(t,s;b)$ can be found in [9]. Let $u_j(t)$, $j = 1,2$, solve (2.2) with initial conditions (2.4). The functions u_j define the Green's function $G(t,s;b)$ [4] to have the form

$$G(t,s;b) = K(s,T) \cdot \begin{cases} \dfrac{u_1(s)u_2(t)}{u_1(T)} & \text{if } 0 \le s \le t \le T \\ \\ \dfrac{u_1(t)u_2(s)}{u_1(T)} & \text{if } 0 \le t \le s \le T, \end{cases} \tag{2.5}$$

where $K(s,T)$ is a negative function. By applying Lemma 2.2 the result is established.||

Note 2.4: The precise form of the exponential function $K(s,T)$ is $-\exp(-\int_s^T a(x)dx)$. However, the form of $G(t,s;b)$, which is now known to be unique under the hypotheses of Theorem 2.3, will be displayed in a somewhat different manner in the work that follows.

In addition to portions of many text books on differential equations, such as [7] and [9], which treat the theory of Green's functions in boundary value theory, there are entire volumes on the subject, such as [6]. A very minimal amount of this theory of Green's functions will now be given.

Assume that Equation 2.2 is disconjugate on $[0, T]$, and let $v_j(t)$, $j = 1,2$, be solutions of (2.2) which satisfy the initial conditions

$$v_j(0) = \delta_{2j}, \quad v_j'(0) = \delta_{1j}, \tag{2.6}$$

where δ_{ij} is the Kroneker delta. Then the Green's function $G(t,s;b)$ of (2.5) can be written, with $p(s)=\exp(-\int_0^s a(x)\,dx)$, as

$$G(t,s;b) = -\frac{p(s)}{v_1(T)} \cdot \begin{cases} v_1(s)[v_1(t)v_2(T) - v_2(t)v_1(T)] \\ \qquad\qquad \text{if} \quad 0 \le s \le t \le T \\ v_1(t)[v_1(s)v_2(T) - v_2(s)v_1(T)] \\ \qquad\qquad \text{if} \quad 0 \le t \le s \le T. \end{cases} \tag{2.7a}$$

Details of this construction can be found in [17].

From (2.7a), it is easily seen that $G_o(t,s)$ has the form

$$G_o(t,s) = \frac{1}{T} \cdot \begin{cases} (T - t)s & \text{if} \quad 0 \le s \le t \le T \\ (T - s)t & \text{if} \quad 0 \le t \le s \le T. \end{cases} \tag{2.7b}$$

Since the functions v_j are, in general, difficult to compute, (2.7a) is much more appealing than (2.7a). Consequently, the use of $G_o(t,s)$ as an integral kernel has definite advantages over the use of $G(t,s;b)$. In particular, the magnitude of $G_o(t,s)$ and its partial derivatives are easily computed. For example,

$$0 \leq G_o(t,s) \leq \frac{T}{4}, \tag{2.8a}$$

$$\int_o^T G_o(t,s)\,ds = \frac{(T-t)t}{2} \leq \frac{T^2}{8}, \tag{2.8b}$$

$$\int_o^T G_o^2(t,s)\,ds = \frac{(T-t)^2 t^2}{3T} \leq \frac{T^3}{48}, \tag{2.8c}$$

$$\int_o^T \left|\frac{\partial}{\partial t} G_o(t,s)\right| ds = \frac{t^2 + (T-t)^2}{2T} \leq \frac{T}{2}. \tag{2.8d}$$

The Green's function $G(t,s;b)$ finds its application to boundary value theory through the linear nonhomogeneous boundary value problem

$$Ly = b(t)y + g(t); \quad y(0) = y(T) = 0, \tag{2.9}$$

where it is assumed that $g(t)$ is continuous on $[0, T]$. The reason is given in the following theorem, which is stated without proof.

Theorem 2.5: If Equation 2.2 is disconjugate on $[0, T]$, then Equation 2.9 has a unique solution $y(t)$ which is given by

$$y(t) = -\int_0^T G(t,s;b)g(s)ds. \qquad (2.10)$$

Furthermore, the first derivative of the solution, $y'(t)$, is given by

$$y'(t) = -\int_0^T \frac{\partial}{\partial t} G(t,s;b)g(s)ds. \qquad (2.11)$$

Because of Theorem 2.5, it is clear why (1.1) is said to be equivalent to (1.4), namely, $h(t,y,y')$ of (1.1) is considered as the nonhomogeneous forcing function added to the zero function in the equation $y'' = 0$, and the kernel $G(t,s;b)$ in (2.10) is replaced by $G_0(t,s)$. In addition to the equivalence of (1.4) and (1.1), straightforward differentiation of (1.4) along with (2.7b) shows that the derivative of the solution to (1.1) is a solution to the integral equation

$$h'(t) = -\int_0^T \frac{\partial}{\partial t} G_0(t,s)h(s,y(s),y'(s))ds + \omega'(t). \qquad (2.12)$$

Let $b_j(t)$, $j = 1,2$, be continuous on $[0, T]$, and denote by $G(t,s;b_j)$, $j = 1,2$, the Green's functions, if they exist, for the linear problems

$$Ly = b_j(t)y; \quad j = 1,2. \qquad (2.13)$$

<u>Theorem 2.6</u>: Suppose Equation 2.13 is disconjugate on

$[0, T]$ for $j = 1$, and that $b_1(t) \leq b_2(t)$ on $[0, T]$. Then $G(t,s;b_j)$, $j=1,2$, exist, are unique, and are continuous and, furthermore,

$$0 \leq G(t,s;b_2) \leq G(t,s;b_1) \tag{2.14}$$

on $[0, T] \times [0, T]$.

Proof: Since (2.13) is disconjugate on $[0, T]$ for $j = 1$ and $b_1(t) \leq b_2(t)$ on $[0, T]$, it follows from the variational principle [7] that (2.13) is also disconjugate on $[0, T]$ for $j = 2$. Thus, by Theorem 2.3 $G(t,s;b_j)$, $j = 1,2$, exist, are unique, are continuous, and are non-negative on $[0, T] \times [0, T]$.

Suppose

$$G(t_o,s_o;b_2) - G(t_o,s_o;b_1) > 0 \tag{2.15}$$

for some $(t_o,s_o) \in [0, T] \times [0, T]$. By continuity in the s-variable, there must exist a $\delta > 0$ such that

$$G(t_o,s;b_2) - G(t_o,s;b_1) > 0$$

whenever $|s - s_o| < \delta$. Define $g(t)$ by

$$g(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq s_o - \delta \\ -\exp\left(\dfrac{1}{\left(\frac{t-t_o}{\delta}\right)^2 - 1}\right) & \text{if } s_o - \delta < t < s_o + \delta \\ 0 & \text{if } s_o + \delta \leq t \leq T. \end{cases} \tag{2.16}$$

Let $u_j(t)$ be the unique solutions, $j = 1,2$, to the boundary value problems

$$Ly = b_j(t)y + g(t); \quad y(0) = y(T) = 0, \quad j = 1,2. \tag{2.17}$$

By (2.10),

$$u_j(t) = -\int_0^T G(t,s;b_j)g(s)\,ds, \qquad j = 1,2, \tag{2.18}$$

and thus $u_j(t) \geq 0$ on $[0, T]$, $j = 1,2$. Let $z(t) = u_2(t) - u_1(t)$. Then $z(t)$ solves the boundary value problem

$$Lz = b_1(t)z + (b_2(t) - b_1(t))u_2; \quad z(0) = z(T) = 0. \tag{2.19}$$

Therefore, z satisfies the differential inequality

$$Lz \geq b_1(t)z, \tag{2.20}$$

which implies that z is a lower solution with respect to solutions of the boundary value problem

$$Ly = b_1(t)y; \quad y(0) = y(T) = 0. \tag{2.21}$$

Thus $z(t) \leq 0$ on $[0, T]$, which is a contradiction to (2.15), since

$$0 < z(t_o) = -\int_{s_o-\delta}^{s_o+\delta} (G(t_o,s;b_2) - G(t_o,s;b_1))g(s)\,ds. \quad ||$$

Note 2.7: References on the theory of sub(super)-functions and lower(upper)-solutions which insure $z(t) \leq 0$ in Theorem 2.6 are given in Chapter 4.

The Green's function $G(t,s;0)$ has a particularly simple form, namely, with $p(s) = \exp(-\int_0^s a(x)\,dx)$,

$$G(t,s;\,0) = p(s)\cdot\begin{cases} v(s)[1 - \frac{v(t)}{v(T)}] & \text{if } 0 \le s \le t \le T \\ v(t)[1 - \frac{v(s)}{v(T)}] & \text{if } 0 \le t \le s \le T, \end{cases} \tag{2.22}$$

where

$$v(t) = \int_0^t \frac{ds}{p(s)}\,. \tag{2.23}$$

<u>Corollary 2.8</u>: If $0 \le b_1(t) \le b_2(t)$ on $[0, T]$, then Equation 2.13 is disconjugate on $[0, T]$ and, moreover,

$$0 \le G(t,s;b_j) \le G(t,s;0), \quad j = 1,2. \tag{2.24}$$

<u>Proof</u>: Equation 2.13 is disconjugate on $[0, T]$ because of the disconjugacy condition (2.3). The proof of (2.24) is clear from Theorem 2.6. ||

The following result is a direct consequence of Theorem 2.6.

<u>Corollary 2.9</u>: If in Equation 2.1 $a(t) \equiv 0$ on $[0, T]$ and $0 \le b_1(t) \le b_2(t)$ on $[0, T]$, then (2.13) is disconjugate on $[0, T]$ and, moreover,

$$0 \le G(t,s;b_j) \le G_0(t,s), \quad j = 1,2. \tag{2.25}$$

The example that follows shows that, in general, neither

$G(t,s;0)$ nor $G_o(t,s)$ dominates the other.

Example 2.10: Let $a(t)$ be a nonzero constant on $[0, T]$, say, $a(t) \equiv a$. Then $v(t)$ defined by (2.23) becomes

$$v(t) = \frac{e^{at} - 1}{a}$$

and $G(t,s;0)$ defined by (2.22) has the form

$$G(t,s;0) = \frac{e^{-as}}{a} \begin{cases} (e^{as} - 1)(1 - \dfrac{e^{at} - 1}{e^{aT} - 1}) & \text{if } 0 \leq s \leq t \leq T \\ (e^{at} - 1)(1 - \dfrac{e^{as} - 1}{e^{aT} - 1}) & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

Therefore, if $a = \ell n\, 2$ and $T = 2$, it follows that

$$G(t,s;0) = \frac{2^{-s}}{3\ell n 2} \begin{cases} (2^s - 1)(4 - 2^t) & \text{if } 0 \leq s \leq t \leq 2 \\ (2^t - 1)(4 - 2^s) & \text{if } 0 \leq t \leq s \leq 2. \end{cases}$$

Thus, at $(t,s) = (1,1/3)$,

$$G(1,1/3;0) = \frac{2^{-1/3}}{3\ell n 2}(2^{1/3} - 1)(4 - 2)$$

$$= \frac{2 - 2^{2/3}}{3\ell n 2} > \frac{1}{6} = G_o(1,1/3).$$

On the other hand, if $a = \ell n 2$, $T = 2$, and $(t,s) = (1,1)$, it follows that

$$G(1,1;0) = \frac{1}{3\ln 2} < \frac{1}{2} = G_o(1,1).$$

In [13], Lees proves a lemma and a theorem which are stated here without proofs.

<u>Lemma 2.11</u>: If $f(t,y)$ has continuous partial derivatives of all orders on $[0, T] \times R$ and satisfies

$$\inf\{f_y(t,y): 0 \leq t \leq 1, \quad -\infty < y < \infty\} = -\eta > -\pi^2, \qquad (2.26)$$

then any solution y to the two-point boundary value problem

$$y'' = f(t,y); \quad y(0) = y(1) = 0 \qquad (2.27)$$

satisfies

$$\|y\|_\infty \leq \frac{\pi}{2(\pi^2-\eta)} \|f(t,0)\|_\infty, \qquad (2.28a)$$

where the norm $\|\cdot\|_\infty$ is defined on any interval I to be

$$\|f\|_\infty = \sup\{|f(t)|: t \in I\}.$$

Moreover, for any $t, \overline{t} \in [0, 1]$,

$$|y(t) - y(\overline{t})| \leq \frac{\pi}{(\pi^2-\eta)} \|f(t,0)\|_\infty |t-\overline{t}|^{1/2}. \qquad (2.29a)$$

<u>Theorem 2.12</u>: If $f(t,y)$ has continuous partial derivatives of all orders on $[0, T] \times R$ and satisfies (2.26), then (2.27) has a unique solution.

A careful analysis of the proofs given in [13] reveals the following facts:

(1) It suffices merely to have the first partial derivatives of $f(t,y)$ with respect to y continuous on $[0, T] \times R$.

(2) Inequalities 2.28a and 2.29a can be replaced respectively by

$$\|y\|_\infty \leq \frac{\pi}{2(\pi^2-\eta)} \|f(t,0)\|_2, \tag{2.28b}$$

$$|y(t) - y(\bar{t})| \leq \frac{\pi}{(\pi^2-\eta)} \|f(t,0)\|_2 \, |t-\bar{t}|^{1/2}, \tag{2.29b}$$

where the norm $\|\cdot\|_2$ is defined on any interval I to be

$$\|f\|_2 = (\textstyle\int_I |f(t)|^2 dt)^{1/2}.$$

(3) If the interval $[0, 1]$ is replaced by $[0, T]$, inequalities 2.28b and 2.29b can be replaced respectively by

$$\|y\|_\infty \leq \frac{\pi T^{3/2}}{2(\pi^2-\eta T^2)} \|f(t,0)\|_2, \tag{2.28c}$$

$$|y(t)-y(\bar{t})| \leq \frac{\pi T^{3/2}}{(\pi^2-\eta T^2)} \|f(t,0)\|_2 \, |t-\bar{t}|^{1/2}, \tag{2.29c}$$

where the norms are with respect to the interval $[0, T]$.

Lemma 2.13: Suppose that in Equation 2.1 $a(t) \equiv 0$ on $[0, T]$ and that there exists an $\eta \in [0, \pi^2/T^2)$ such that $b(t) \geq -\eta > -\pi^2/T^2$ on $[0, T]$. Then (2.2) is disconjugate on $[0, T]$.

Proof: If $y(t)$ solves Equation 2.2 and has two zeros on $[0, T]$, say at t_1, t_2, then y solves the boundary value problem

$$y'' = b(t)y; \quad y(t_1) = y(t_2) = 0.$$

By changing the independent variable and applying (2.28c), it follows that $y(t) \equiv 0$ on $[0, T]$.||

By using Theorem 2.6 and Lemma 2.13, the result of Lemma 2.11 can be improved as follows.

Theorem 2.14: If $f_y(t,y)$ is continuous on $[0, T] \times R$ and satisfies

$$\inf\{f_y(t,y): 0 \leq t \leq T, \ -\infty < y < \infty\} \geq -\eta > -\pi^2/T^2, \qquad (2.30)$$

then there exists a unique solution y to the boundary value problem

$$y'' = f(t,y); \quad y(0) = y(T) = 0, \qquad (2.31)$$

and, moreover,

$$||y||_\infty \leq K|| f(t,0) ||_2, \qquad (2.32)$$

where

$$K = \frac{\sqrt{2}}{4\eta^{3/4}\cos(\frac{T}{2}\sqrt{\eta})}(T\sqrt{\eta} - \sin(T\sqrt{\eta}))^{1/2}. \qquad (2.33)$$

Also, for any $t,\bar{t} \in [0, T]$,

$$|y(t) - y(\bar{t})| \leq \frac{\pi T^{3/2}}{(\pi^2 - \eta T^2)} \|f(t,0)\|_2 |t-\bar{t}|^{1/2}. \qquad (2.34)$$

Proof: Existence and uniqueness are established by making a change in the independent variable in Theorem 2.12. From the quasi-linearization 1.7 of the boundary value problem 2.31, it follows that

$$B(t;u) \geq -\eta > -\pi^2/T^2$$

for all continuous functions $u(t)$. Let $b(t) = B(t;y)$ where y is the unique solution of (2.31). Then $b(t) \geq -\eta > -\pi^2/T^2$, and y solves the boundary value problem

$$y'' = b(t)y + f(t,0); \quad y(0) = y(T) = 0.$$

Therefore,

$$y(t) = -\int_0^T G(t,s;b)f(s,0)ds, \qquad (2.35)$$

where $G(t,s;b)$ is the Green's function for Equation 2.2 with $a(t) \equiv 0$ on $[0, T]$. By Equations 2.14 and 2.35, it follows that

$$|y(t)| \leq \max_{0\leq t\leq T} \|G(t,s;-\eta)\|_2 \|f(t,0)\|_2,$$

where $G(t,s;-\eta)$ is the Green's function for (2.2) with $a(t) \equiv 0$ on $[0, T]$ and $b(t) \equiv -\eta$ on $[0, T]$, and

$$\|G(t,\cdot;-\eta)\|_2 = (\int_0^T G^2(t,s;-\eta)ds)^{1/2}.$$

The proof of Inequality (2.32) as well as the proof that (2.32) is sharper than (2.28c) will be complete if it can be shown that

$$\max_{0\leq t\leq T} \|G(t,s;-\eta)\|_2 = K < \frac{\pi T^{3/2}}{2(\pi^2-\eta T^2)},$$

where K is defined by (2.33). A major portion of this is established in the following lemma.

Lemma 2.15: If $m(t) = \|G(t,\cdot;-\eta)\|_2$, then $m(t) \leq m(T/2)$ on $[0, T]$.

Proof: It follows from (2.7a) that

$$G(t,s;-\eta) = \frac{1}{\sqrt{\eta}\ \sin(T\sqrt{\eta})} \cdot \begin{cases} \sin(s\sqrt{\eta})\ \sin((T-t)\sqrt{\eta}) \\ \qquad \text{if } 0 \leq s \leq t \leq T \\ \sin(t\sqrt{\eta})\ \sin((T-s)\sqrt{\eta}) \\ \qquad \text{if } 0 \leq t \leq s \leq T, \end{cases} \tag{2.36a}$$

and, therefore,

$$m^2(t) = \frac{1}{\eta\sin^2(T\sqrt{\eta})}\,[\sin^2((T-t)\sqrt{\eta})(t/2 - \frac{\sin(2t\sqrt{\eta})}{4\sqrt{\eta}})$$

$$+ \sin^2(t\sqrt{\eta})(T/2 - t/2 - \frac{\sin(2(T-t)\sqrt{\eta})}{4\sqrt{\eta}})]. \qquad (2.36b)$$

Let $\sqrt{\eta} = c$ and define $f(t)$ by

$$f(t) = \sin^2((T-t)c)(t/2 - \frac{\sin 2tc}{4c})$$

$$+ \sin^2(tc)(T/2 - t/2 - \frac{\sin(2(T-t)c)}{4c}).$$

Then

$$f'(t) = \frac{1}{2}\sin^2((T-t)c)(1 - \cos(2tc))$$

$$- \frac{1}{2}\sin^2(tc)(1 - \cos(2(T-t)c))$$

$$- \frac{tc}{2}\sin(2(T-t)c) + \frac{(T-t)c}{2}\sin(2tc)$$

$$= \frac{c}{2}[(T-t)\sin(2tc) - t\sin(2(T-t)c)].$$

The last equation follows since the first two terms of $f'(t)$ add up to zero by using the identity $2\sin^2\theta = 1 - \cos 2\theta$. Thus, $f'(T/2) = 0$. To obtain $m(t) \leq m(T/2)$, it suffices to show $f'(t) \geq 0$ on $[0, T/2]$ and $f'(t) \leq 0$ on $[T/2, T]$. Let $t \in [0, T/2]$. If $\sin(2(T-t)c) \leq 0$, it follows that $f'(t) \geq 0$, so, suppose that $\sin(2(T-t)c) > 0$.

Since $0 \le c^2 < \pi^2/T^2$, it follows that

$$1 - \frac{4c^2t^2}{j^2\pi^2} \ge 0, \qquad j = 1,2,3,\cdots,$$

and also that

$$1 - \frac{4c^2(T-t)^2}{j^2\pi^2} \ge 0, \qquad j = 1,2,3,\cdots. \tag{2.36c}$$

Note that (2.36c) is true for $j = 1$ since $\sin(2(T-t)c) \ge 0$ implies that $2(T-t)c \le \pi$. Furthermore, it follows that

$$1 - \frac{4c^2t^2}{j^2\pi^2} \ge 1 - \frac{4c^2(T-t)^2}{j^2\pi^2}, \quad j = 1,2,3,\cdots,$$

and, thus, since $\sin x = x \prod_{j=1}^{\infty} (1 - \frac{x^2}{j^2\pi^2})$,

$$\begin{aligned} f'(t) &= \frac{c}{4} \Big[(T-t)2tc \prod_{j=1}^{\infty} (1 - \frac{4c^2t^2}{j^2\pi^2}) \\ &\qquad - t(2(T-t)c) \prod_{j=1}^{\infty} (1 - \frac{4c^2(T-t)^2}{j^2\pi^2})\Big] \\ &= \frac{c^2t(T-t)}{2} \Big[\prod_{j=1}^{\infty} (1 - \frac{4c^2t^2}{j^2\pi^2}) - \prod_{j=1}^{\infty} (1 - \frac{4c^2(T-t)^2}{j^2\pi^2}) \Big] \\ &\ge 0. \end{aligned}$$

The fact that $f'(t) \le 0$ for $T/2 \le t \le T$ follows since $f'(t) = -f'(T-t)$. Thus, Lemma 2.15 is proved. ||

It follows by direct substitution into $m(t)$ that $m(T/2) = K$. In order to show that

$$K < \frac{\pi T^{3/2}}{2(\pi^2-\eta T^2)}, \qquad 0 \le \eta < \pi^2/T^2, \tag{2.37}$$

it suffices to show that

$$K^2 < \frac{\pi^2 T^3}{4(\pi^2-\eta T^2)^2}.$$

Putting $T\sqrt{\eta} = 2X$, this reduces to

$$X \sec^2 x - \tan x < \frac{8\pi^2 x^3}{(\pi^2-4x^2)^2}; \qquad 0 \le x < \pi/2. \tag{2.38a}$$

By long division, one gets

$$\frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2} = \sum_{n=2}^{\infty} \frac{(n-1)2^{2n-1}x^{2n-1}}{\pi^{2n-2}} \qquad 0 \le x < \pi/2.$$

Since $x \sec^2 x - \tan x = x + \tan x\,(x \tan x - 1)$, and since

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_n x^{2n-1} \qquad 0 \le x < \pi/2, \tag{2.38b}$$

where B_n, the n^{th} Bernoulli number, is

$$B_n = \frac{2(2n)!}{\pi^{2n}(2^{2n} - 1)} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n},$$

We compute the Cauchy product of $\tan x(x \tan x)$ and add $x - \tan x$ to get the series expansion of $x \sec^2 x - \tan x$ as

$$\sum_{n=2}^{\infty} \left[\sum_{j=1}^{n-1} \frac{2^{2j}(2^{2j}-1)}{(2j)!} B_j \cdot \frac{2^{2(n-j)}(2^{2(n-j)}-1)}{(2(n-j))!} B_{n-j} - \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \right] x^{2n-1}.$$

Substituting for B_j, B_{n-j}, and B_n and collecting terms gives

$$x \sec^2 x - \tan x = \sum_{n=2}^{\infty} \left[\sum_{j=1}^{n-1} \frac{4 \cdot 2^{2n}}{\pi^{2n}} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2j} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-j)} - \frac{2 \cdot 2^{2n}}{\pi^{2n}} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} \right] x^{2n-1}. \qquad (2.38c)$$

From (2.38b,c) it follows that a sufficient condition for inequality 2.38a is that corresponding coefficients of x^{2n-1} satisfy

$$\frac{4 \cdot 2^{2n}}{\pi^{2n}} \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2j} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-j)} < \frac{2 \cdot 2^{2n}}{\pi^{2n}} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} + \frac{(n-1)2^{2n-1}}{\pi^{2n-2}}$$

for $n = 2, 3, 4, \cdots$. It thus suffices to show that

$$2 \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2j} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-j)} < \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} + \frac{(n-1)\pi^2}{4}.$$

Now

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^2 = \frac{\pi^2}{8}, \quad \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^4 = \frac{\pi^4}{96},$$

and thus

$$2\sum_{j=1}^{n-1}\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2j} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-j)} = \frac{4\pi^2}{8}\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-1)}$$

$$+ 2\sum_{j=2}^{n-2}\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2j} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-j)}.$$

It is enough to show that

$$\frac{\pi^2}{2}\sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2(n-1)} + \frac{2(n-3)\pi^8}{(96)^2} \leq \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} + \frac{(n-1)\pi^2}{4},$$

or

$$\frac{\pi^4}{16} + \frac{(n-1)\pi^8}{4608} - \frac{\pi^8}{2304} < \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} + \frac{(n-1)\pi^2}{4},$$

which is equivalent to

$$(n-1)\left[\frac{\pi^8}{4608} - \frac{\pi^2}{4}\right] < \sum_{k=1}^{\infty} \left(\frac{1}{2k-1}\right)^{2n} + \frac{\pi^8}{2304} - \frac{\pi^4}{16}.$$

It therefore suffices to show that

$$(n-1)\left[\frac{\pi^8}{4608} - \frac{\pi^2}{4}\right] < 1 + \frac{\pi^8}{2304} - \frac{\pi^4}{16}$$

$$\simeq -.9697894.$$

Since $\frac{\pi^8}{4608} - \frac{\pi^2}{4} \simeq -.40826783$, the result follows for $n \geq 4$.

The cases $n = 2,3$ are easily shown separately. $\|$

Lees' result (Lemma 2.11) can be improved even more by applying Theorem 2.14 to the problem

$$Ly = f(t,y); \qquad y(0) = y(T) = 0, \tag{2.39}$$

where $a(t)$ is not assumed to be identically zero.

<u>Corollary 2.16</u>: If $f_y(t,y)$ is continuous on $[0, T] \times R$, $a(t) \in C'[0, T]$, and

$$\frac{a^2(t)}{4} - \frac{a'(t)}{2} + \inf\{f_y(t,y): 0 \le t \le T, \quad -\infty < y < \infty\}$$
$$\ge -\eta > -\pi^2/T^2, \tag{2.40}$$

then there exists a unique solution y to (2.39) and, moreover,

$$\|Z\|_\infty \le K \|F(t,0)\|_2 \tag{2.41}$$

and, for any $t,\bar{t} \in [0, T]$,

$$|Z(t) - Z(\bar{t})| \le \pi T^{3/2} C|t - \bar{t}|^{1/2}/(\pi^2 - \eta T^2) \tag{2.42}$$

where

K is defined by (2.33),

$A(t) = \exp[-(1/2)\int_0^t a(s)ds]$,

$Z(t) = A(t)y(t)$,

$F(t,z) = [\frac{1}{4} a^2(t) - \frac{1}{2} a'(t)]Z + A(t) f(t, Z/A(t))$,

$C = \max\{\|F(t,0)\|_2 : 0 \leq t \leq T\}$.

Proof: Equation 2.39 is equivalent to the problem

$$Z'' = F(t, Z); \qquad Z(0) = Z(T) = 0$$

and Theorem 2.14 applies to this equation. ||

Note 2.17: When $\eta = 0$, k in (2.33) is indeterminant. Putting $x = T\sqrt{\eta}$, we obtain $k^2 = \frac{T^3}{4x^3}\left(\frac{x - \sin x}{1 + \cos x}\right)$, and, applying L'Hospital's rule as $x \to 0^+$, $K^2 = \frac{T^3}{48}$ when $\eta = 0$.

III. SUCCESSIVE APPROXIMATIONS

Recall the definitions (1.6a-c) and the quasi-linearization (1.7) of the boundary value problem (1.1). If a sequence of successive approximations can be shown to converge to a solution $u(t)$ of (1.7), then by (1.6a), $u(t) + \omega(t)$ will solve (1.1).

Suppose that $h(t,y,y')$ has continuous second partial derivatives on $[0, T] \times R^2$, that $u(t)$ solves (1.7), and that

$$Z(t) = u(t)\exp(-\tfrac{1}{2}\int_0^t A(s;u(s))ds), \tag{3.1a}$$

$$g(t) = h(t,\omega,\omega')\exp(-\tfrac{1}{2}\int_0^t A(s;u(s))ds). \tag{3.1b}$$

It can then be shown [7] that (1.7) is equivalent to the quasi-linear boundary value problem

$$Z'' = [\tfrac{1}{4} A^2(t;u) - \tfrac{1}{2} \tfrac{d}{dt} A(t;u) + B(t;u)]Z + g(t);$$
$$Z(0) = Z(T) = 0. \tag{3.2}$$

The aim in the work that follows is to obtain existence of a solution to (1.7) by making assumptions on $A(t;u)$ and $B(t;u)$ for functions $u(t)$ contained in some class of functions S, and then by applying Corollary 2.16, the Ascoli Theorem, and the Schauder Fixed-Point Theorem. With somewhat stronger assumptions, uniqueness is obtained within the specified class S. Finally, with more assumptions, the unique solution within the specified class S is shown to be the limit of a sequence of successive approximations.

In view of (2.32) and (2.41), along with the objective of using the Ascoli Theorem, a possible class of functions S to consider is the set of those functions which are merely bounded by some constant or by some constant multiple of a positive continuous function on $[0, T]$. However, the coefficient of Z in (3.2) involves not only t and $u(t)$, but also $u'(t)$ and $u''(t)$, since

$$\begin{aligned}\frac{d}{dt} A(t;u) = \int_0^1 [&h_{y't}(t,\xi u+\omega,\xi u'+\omega') \\ &+ h_{y'y}(t,\xi u+\omega,\xi u'+\omega')(\xi u'+\omega') \\ &+ h_{y'y'}(t,\xi u+\omega,\xi u'+\omega)(\xi u'')]d\xi. \qquad (3.3)\end{aligned}$$

In order to avoid this difficulty, the remainder of Chapter 3 will deal with the boundary value problem (1.1) with the assumption that $h(t,y,y')$ is linear in y', i.e.,

$$Ly = f(t,y); \qquad y(0) = A, \quad y(T) = B, \qquad (3.4)$$

where it is assumed that $a(t)$ of (2.1) is continuously differentiable on $[0, T]$ and that $f_y(t,y)$ is continuous on $[0, T] \times R$. With these assumptions, equations (1.6a-c) have the respective forms

$$u(t) = y(t) - \omega(t),$$

$$A(t;u) = a(t),$$

$$B(t;u) = \int_o^1 f_y(t,\xi u+\omega)d\xi,$$

and the quasi-linearization (1.7) becomes

$$Lu = B(t;u)u + a(t)\omega'(t) + f(t,\omega); \quad u(0) = u(T) = 0. \tag{3.5}$$

Other simplifications due to this assumption are that (3.1a-b) become, respectively,

$$Z(t) = u(t)\exp(-\tfrac{1}{2}\int_o^t a(s)ds), \tag{3.6a}$$

$$g(t) = (a(t)\omega'(t) + f(t,\omega))\exp(-\tfrac{1}{2}\int_o^t a(s)ds). \tag{3.6b}$$

For any fixed $\eta \in [0, \pi^2/T^2)$, define the class of functions

$$S = \{y \in C[0, T]: y(0) = y(T) = 0, |y(t)| \leq K\,||g||_2 \exp(\tfrac{1}{2}\int_o^t a(s)ds)\}, \tag{3.7}$$

where $g(t)$ is defined by (3.6b) and, as in (2.33),

$$K = \frac{\sqrt{2}}{4\eta^{3/4}} \frac{(T\sqrt{\eta} - \sin(T\sqrt{\eta}))^{1/2}}{\cos(\frac{T\sqrt{\eta}}{2})}.$$

<u>Note 3.1</u>: If there exists an $\eta \in [0, \pi^2/T^2)$ such that

$$\frac{a^2(t)}{4} - \frac{a'(t)}{2} + \int_o^1 f_y(t,\xi u+(1-\xi)v+\omega)d\xi \geq -\eta > -\pi^2/T^2 \tag{3.8}$$

for all $u,v \in S$, then, by choosing $v \equiv 0$, it follows that

$$\frac{a^2(t)}{4} - \frac{a'(t)}{2} + B(t;u) \geq -\eta > -\pi^2/T^2 \tag{3.9}$$

for all $u \in S$.

Theorem 3.2: If there exists an $\eta \in [0, \pi^2/T^2)$ such that (3.9) holds, then (3.5) has a solution $u_1(t)$ which remains in S. Moreover, if (3.8) holds, then $u_1(t)$ is unique within S.

Proof: Define the map $\Gamma: S \to C^2[0, T]$ by $\Gamma u = v$, where

$$Lv = B(t;u)v + a(t)\omega'(t) + f(t,\omega); \quad v(0) = v(T) = 0. \tag{3.10}$$

For fixed $u \in S$, assumption (3.9) guarantees that the equation

$$y'' = [\tfrac{1}{4} a^2(t) - \tfrac{1}{2} a'(t) + B(t;u)]y \tag{3.11}$$

is disconjugate on $[0, T]$. Therefore, by Theorem 2.5, there exists a unique solution $y \in c^2[0, T]$ to the boundary value problem

$$y'' = [\tfrac{1}{4} a^2(t) - \tfrac{1}{2} a'(t) + B(t;u)]y + g(t); \quad y(0)=y(T)=0. \tag{3.12}$$

Hence, by means of (3.6a), there exists a unique solution v to (3.10). Thus, Γ is well defined. Moreover, inequality (2.41) of Corollary 2.16 applied to the boundary value problem (3.12) insures that

$$\|e^{-\frac{1}{2}\int_0^t a(s)ds} v\|_\infty \leq K\,\|g\|_2,$$

and it follows from the definition of S that $v \in S$. Therefore, $\Gamma\colon S \to S$ and, necessarily, $\Gamma(S)$ is uniformly bounded.

Let S have the topology of uniform convergence. It can be shown by an elementary argument that the solution of (3.10) depends continuously upon the coefficient function $B(t;u)$ which, in turn, depends continuously upon $u(t)$. Therefore, Γ is continuous on S. The equicontinuity of $\Gamma(S)$ follows from (2.42) applied to the boundary value problem (3.12). By Ascoli's Theorem, $\overline{\Gamma(S)}$, the closure of $\Gamma(S)$, is compact in $C[0, T]$ and, by the Schauder Fixed-Point Theorem, Γ has a fixed point $u_1(t)$. By the definition of Γ, it follows that $u_1(t)$ solves (3.5) and, necessarily, $u_1 \in S$.

Suppose Γ has two fixed points in S, $u_j(t)$, $j = 1,2$. Then $y_j = u_j + \omega$ solves the problem

$$Ly_j = f(t,y_j); \qquad y(0) = A, \quad y(T) = B, \quad j = 1,2,$$

and $u = u_1 - u_2$ solves the problem

$$Lu = f(t,y_1) - f(t,y_2); \qquad u(0) - u(T) = 0.$$

Therefore, u solves the problem

$$Lu = \left(\int_0^1 f_y(t,\xi u_1+(1-\xi)u_2+\omega)d\xi\right)u; \quad u(0) = u(T) = 0$$

and, consequently, $Z(t) = u(t)\exp(-\frac{1}{2}\int_0^t a(s)ds)$ solves the problem

$$Z'' = \left(\int_0^1 f_y(t,\xi u_1+(1-\xi)u_2+\omega)d\xi+\frac{1}{4}a^2(t)-\frac{1}{2}a'(t)\right)Z;$$
$$Z(0) = Z(T) = 0. \qquad (3.13)$$

Therefore, assumption 3.8 implies that (3.13) has a unique solution, namely, $Z \equiv 0$. Therefore, $u \equiv 0$, and uniqueness within S is established. ||

Examples are quite easy to find which show that the hypotheses of Theorem 3.2 are not sufficient to prevent solutions for (3.5) outside of S, even if it is possible to choose η equal to zero.

<u>Example 3.3</u>: In (3.4), let $T = 2$, $a(t) \equiv 0$ on $[0, 2]$, $A = B = 1$, and $f(t,y) = y(1 + \frac{1}{4}y)$. If $\eta = 2$, then

$$S = \{y \in C[0, 2]: y(0) = y(2) = 0, \quad |y(t)| \leq \frac{5K\sqrt{2}}{4}\},$$

where

$$K = \frac{\sqrt{2}}{4}\left(\sec^2(\sqrt{2}) - \frac{\sqrt{2}}{2}\tan(\sqrt{2})\right)^{1/2}.$$

The following is a sufficient condition for (3.8) to be satisfied, namely,

$$f_y(t,u+\omega) = \frac{u+3}{2} \geq -2 > -\pi^2/4$$

for all $u \in S$. Therefore, although there exists a unique solution which remains in S, it is shown in [1] that there is indeed another solution to the problem which does not remain in S. If, for the same problem, $\eta = 0$ is chosen, then, by Note 2.17 with $k = \frac{1}{\sqrt{6}}$,

$$S = \{y \in C[0,\ 2]\colon\ y(0) = y(2) = 0,\quad |y| \leq \frac{5\sqrt{3}}{12}\},$$

and (3.8) is again satisfied since

$$f_y(t,u+\omega) = \frac{u+3}{2} \geq 0 > -\pi^2/4$$

for all $u \in S$.

Note 3.4: Example 3.3 illustrates that, although global uniqueness is not guaranteed by Theorem 3.2 for all boundary value problems (3.4), nevertheless, existence and/or uniqueness within the set S is established for classes of problems (3.4) which cannot be handled by other means. In particular, no Lipschitz condition such as (1.5a-b) is assumed, and no lower bound for $f_y(t,y)$ is assumed outside of the class S as in [13].

Note 3.5: It is conjectured that (3.9) is sufficient to guarantee uniqueness within S of a solution to (3.5).

For the sake of simplicity of notation, if the Green's function $G(t,s;B(t;u_k))$ defined by (2.7a) exists for the indexed function $u_k(t)$, $k \neq 0$, it will henceforth be denoted by $G_k(t,s)$. By using this notation, successive approximations for the solution of the boundary value problem (3.5) can be defined by $\Gamma u_k = u_{k+1}$ which, in integral form, becomes

$$u_{k+1}(t) = -\int_0^T G_k(t,s)(a(s)\omega'(s) + f(s,\omega(s)))ds;$$
$$k = 1,2,3,\cdots. \qquad (3.14)$$

The following lemma illustrates the manner in which the comparison theorem for Green's functions (Theorem 2.6) will be used to obtain the unique solution of (3.5) via the successive approximations defined in (3.14).

<u>Lemma 3.6</u>: Suppose that $Ly = B(t;u)y$ is disconjugate on $[0, T]$ for all $u \in S$. Then, for any $u_j \in S$, $j = 1,2$, with $u_1(t) \leq u_2(t)$ on $[0, T]$, if $B(t;u)$ is nondecreasing [nonincreasing] in u for all $u \in S$, it follows that $0 \leq G_2(t,s) \leq G_1(t,s)$ $[0 \leq G_1(t,s) \leq G_2(t,s)]$ on $[0, T] \times [0, T]$.

<u>Proof</u>: Assume that $B(t;u)$ is nondecreasing in u for all $u \in S$. Since $u_1 \leq u_2$, it follows that $B(t;u_1) \leq B(t;u_2)$. The disconjugacy assumption guarantees that $G_j(t,s)$, $j = 1,2$, exist, are unique, are continuous, and are nonnegative on $[0, T] \times [0, T]$. Therefore, by Theorem 2.6, $0 \leq G_2(t,s) \leq G_1(t,s)$. The second case is analogous. ||

Theorem 3.7: Assume the hypotheses of Note 3.1, let $\theta(t) = a(t)\omega'(t) + f(t,\omega(t))$, and let the sequence of iterates $\{u_k(t)\}$ be defined by (3.14) for $k \geq 2$ where $u_1(t) \equiv 0$. Let $v(t)$ denote the unique solution of (3.5) remaining in S.

(i) If $\theta(t) \geq 0$ on $[0, T]$ and $B(t,u)$ is nondecreasing in u for all $u \in S$, then $\{u_k(t)\}$ converges monotonically downward to $v(t)$.

(ii) If $\theta(t) \leq 0$ on $[0, T]$ and $B(t;u)$ is nonincreasing in u for all $u \in S$, then $\{u_k(t)\}$ converges monotonically upward to $v(t)$.

Proof: For any fixed $u \in S$, assumption 3.8 implies the disconjugacy of the equation

$$y'' = [\tfrac{1}{4} a^2(t) - \tfrac{1}{2} a'(t) + B(t;u)]y.$$

Therefore, the Green's function $G(t,s;B(t:u))$ exists, is unique, is continuous, and is nonnegative on $[0, T] \times [0, T]$ and, thus, the successive approximations of (3.14) are well defined. The monotone convergence will be shown for case (i), and case (ii) then follows analagously.

Let $G(t,s)$ be the Green's function for the problem

$$Ly = B(t;v)y, \tag{3.15a}$$

where $v(t)$ is the unique solution of (3.5) which remains in S. Then

$$v(t) = -\int_0^T G(t,s)\theta(s)ds \leq 0 = u_1(t).$$

By Lemma 3.6, $0 \leq G_1(t,s) \leq G(t,s)$ on $[0, T] \times [0, T]$ and, thus,

$$v(t) = -\int_0^T G(t,s)\theta(s)ds$$

$$\leq -\int_0^T G_1(t,s)\theta(s)ds = u_2(t) \leq 0 = u_1(t).$$

Assume that $v(t) \leq u_{k+1}(t) \leq u_k(t)$ for some k. By using Lemma 3.6 twice, it follows that $v(t) \leq u_{k+2}(t) \leq u_{k+1}(t)$ and, thus, $\{u_k(t)\}$ converges monotonically downward. Since the limit can be shown to solve (3.5), uniqueness within S guarantees that this limit must be $v(t)$. ||

Theorem 3.7 gives rise to a natural question, namely, what happens if either (i) $\theta(t) \leq 0$ and $B(t;u)$ is nondecreasing in u for all $u \in S$, or (ii) $\theta(t) \geq 0$ and $B(t;u)$ is nonincreasing in u for all $u \in S$? The answer for each case is not as satisfying as it is for the cases considered in Theorem 3.7. The following theorem and example provide these answers.

<u>Theorem 3.8</u>: Assume the hypotheses of Note 3.1, let $\theta(t) = a(t)\omega'(t) + f(t,\omega(t))$, and let the sequence of iterates $\{u_k(t)\}$ be defined by (3.14) for $k \geq 2$ where $u_1(t) \equiv 0$. Let $v(t)$ denote the unique solution of (3.5) remaining in S.

(i) If $\theta(t) \leq 0$ on $[0, T]$ and $B(t;u)$ is nondecreasing in u for all $u \in S$, then $\{u_{2k+1}(t)\}$ converges monotonically upward, $\{u_{2k}(t)\}$ converges monotonically downward, and $0 \leq u_{2k+1}(t) \leq v(t) \leq u_{2k}(t)$.

(ii) If $\theta(t) \geq 0$ on $[0, T]$ and $B(t;u)$ is nonincreasing in u for all $u \in S$, then $\{u_{2k}(t)\}$ converges monotonically upward, $\{u_{2k+1}(t)\}$ converges monotonically downward, and $u_{2k}(t) \leq v(t) \leq u_{2k+1}(t) \leq 0$.

Proof: Case (i) will be proved, and case (ii) will follow analogously.

The unique solution $v(t)$ of (3.5) which remains in S satisfies

$$v(t) = -\int_0^T G(t,s)\,\theta(s)ds \geq 0 = u_1(t).$$

By Lemma 3.6, $0 \leq G(t,s) \leq G_1(t,s)$ on $[0, T] \times [0, T]$ and, thus,

$$0 \leq v(t) \leq -\int_0^T G_1(t,s)\,\theta(s)ds = u_2(t).$$

Assume that $v(t) \leq u_{2k}(t)$ for some k. By using Lemma 3.6 twice, it follows that

$$u_{2k+1}(t) \leq v(t) \leq u_{2k+2}(t).$$

These inequalities, along with the fact that

$$0 = u_1(t) \leq -\int_0^T G_2(t,s)\,\theta(s)ds = u_3(t),$$

establish the monotonic convergence of $\{u_{2k+1}(t)\}$ and

$\{u_{2k}(t)\}$, respectively, upward and downward with $0 \leq u_{2k+1}(t) \leq v(t) \leq u_{2k}(t)$. ||

It might be conjectured that the sequences $\{u_{2k+1}(t)\}$ and $\{u_{2k}(t)\}$ in Theorem 3.8 must converge to the solution $v(t)$. This is, however, not true in general.

<u>Note 3.9</u>: If another assumption is added in Theorem 3.8, it can be shown that $\{u_{2k+1}(t)\}$ and $\{u_{2k}(t)\}$ converge to $v(t)$. This assumption is that if $y,z \in S$ are solutions to the respective problems

$$Ly = B(t;z)y + \theta(t); \qquad y(0) = y(T) = 0,$$

$$Lz = B(t;y)z + \theta(t); \qquad z(0) = z(T) = 0,$$

then $y(t) \equiv z(t)$. To see how this assumption applies to case (i) of Theorem 3.8, let $v_*(t)$ be the limit of $\{u_{2k+1}(t)\}$ and $v^*(t)$ be the limit of $\{u_{2k}(t)\}$. Then $v_*(t)$ and $v^*(t)$ solve the respective problems

$$Lv_* = B(t;v^*)v_* + \theta(t); \qquad v_*(0) = v_*(T) = 0,$$

$$Lv^* = B(t;v_*)v^* + \theta(t); \qquad v^*(0) = v^*(T) = 0.$$

Therefore, $v_*(t) \equiv v^*(t)$ and, necessarily, $v_*(t) \equiv v(t) \equiv v^*(t)$.

The following example illustrates that without some added assumption, such as in Note 3.9, the sequences

$\{u_{2k+1}(t)\}$ and $\{u_{2k}(t)\}$ need not converge to $v(t)$.

<u>Example 3.10</u>: Choose $\alpha > 0$ and let $u_1(t) \equiv 0$, let $u_2(t)$ solve the boundary value problem

$$u_2'' = u_2 - \alpha; \qquad u_2(0) = u_2(T) = 0,$$

let $k(t)$ be a continuous function with $k(0) = k(T) = 1$ and $k(t) > 1$ on $(0, T)$, and let $u_3(t)$ solve the boundary value problem

$$u_3'' = k(t)u_3 - \alpha; \qquad u_3(0) = u_3(T) = 0.$$

Then $0 \lneq u_3(t) \lneq u_2(t)$. Define $B(t;u)$ by

$$B(t;u) = \begin{cases} k(t) & \text{for } u \geq u_2(t) \\ (k(t)-1)\left(\dfrac{u - u_3(t)}{u_2(t)-u_3(t)}\right) + 1 & \text{for } u_3(t) < u < u_2(t) \\ 1 & \text{for } u \leq u_3(t) \end{cases}$$

Then $B(t;u) \geq 0$ for all u and $B(t;u)$ is nondecreasing in u for all u. Let $f(t,y) = yB(t;y) - \alpha$ and consider the problem $Ly = f(t,y)$, where $a(t) \equiv 0$, with zero boundary conditions, i.e.,

$$y'' = f(t,y); \qquad y(0) = y(T) = 0. \tag{3.15b}$$

By construction, the hypotheses of (i) in Theorem 3.8 are satisfied with $\eta = 0$. Then

$$u_2'' = u_2 - \alpha \leq k(t)u_2 - \alpha = f(t,u_2);$$
$$u_2(0) = u_2(T) = 0,$$

and

$$u_3'' = k(t)u_3 - \alpha \geq u_3 - \alpha = f(t,u_3);$$
$$u_3(0) = u_3(T) = 0.$$

Therefore, $u_2(t)$ and $u_3(t)$ are upper and lower solutions, respectively, for the problem 3.15b. Thus, there exists a solution $y(t)$ to (3.15b) such that $u_3(t) \leq y(t) \leq u_2(t)$ [10]. Since $u_3(t) \not\equiv u_2(t)$, it follows that $u_3(t) \underset{\not\equiv}{<} y(t) \underset{\not\equiv}{<} u_2(t)$. By the construction of $B(t;u)$, $u_{2k+1}(t) \equiv u_3(t)$ and $u_{2k}(t) \equiv u_2(t)$ for all $k = 1,2,3,\cdots$. Thus, by the uniqueness of solutions within S, it follows that $u_{2k+1}(t) \underset{\not\equiv}{<} y(t) \underset{\not\equiv}{<} u_{2k}(t)$. A similar example can be constructed for case (ii) of Theorem 3.8.

The assumption described in Note 3.9 is not impossible to verify. For instance, consider the following example, which we know <u>a priori</u> has a unique solution.

<u>Example 3.11</u>: Consider the problem

$$y'' = \alpha(t)y^3 + \beta(t)y^2 + \gamma(t)y + \phi(t);$$
$$y(0) = y(T) = 0, \qquad (3.15c)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are continuous and nonnegative on $[0, T]$ and $\phi(t)$ is continuous and nonpositive on $[0, T]$. Since $y(t) \equiv 0$ is a lower solution for (3.15c), it follows that the solution of (3.15c) is nonnegative and, thus, with

$$B(t;u) = \begin{cases} \alpha(t)u^2 + \beta(t)u + \gamma(t) & \text{for} \quad u \geq 0 \\ \\ \gamma(t) & \text{for} \quad u < 0 \end{cases}$$

consider the problem $y'' = B(t;y)y + \phi(t)$; $y(0) = y(t) = 0$. Thus, it suffices to choose an $\eta \geq 0$ such that $B(t;u) = \alpha(t)u^2 + \beta(t)u + \gamma(t) \geq -\eta > -\pi^2/T^2$ for all $u \in S \cap \{u\colon u \geq 0\}$ and, thus, the modified $B(t;u) \geq -\eta > -\pi^2/T^2$ for all $u \in S$. Since $\beta(t) \geq 0$, it follows that $\beta(t) \geq -2\alpha(t)u$ for all $u \in S \cap \{u\colon u \geq 0\}$. This implies that $B(t;u)$ is nondecreasing in u for all $u \in S \cap \{u\colon u \geq 0\}$ and, thus, the modified $B(t;u)$ is nondecreasing in u for all $u \in S$.

Now suppose that $y, z \in S \cap \{u\colon u \geq 0\}$ such that

$$y'' = [\alpha(t)z^2 + \beta(t)z + \gamma(t)]y + \phi(t);$$
$$y(0) = y(T) = 0,$$

and

$$z'' = [\alpha(t)y^2 + \beta(t)y + \gamma(t)]z + \phi(t);$$
$$z(0) = z(T) = 0.$$

Let $D(t) = y(t) - z(t)$. Then

$$D'' = (\gamma(t) - \alpha(t)yz)D; \qquad D(0) = D(T) = 0.$$

Thus, a sufficient condition for the hypothesis of Note 3.9 to be satisfied, that is, for $D(t)$ to be identically zero, is that

$$\gamma(t) > \alpha(t)K^2 \|\phi\|_2^2 - \pi^2/T^2, \tag{3.15d}$$

where the constant K, which depends upon T and η, is defined by (2.33), i.e.,

$$K = \frac{\sqrt{2}}{4\eta^{3/4}\cos(\frac{T\sqrt{\eta}}{2})}(T\sqrt{\eta} - \sin(T\sqrt{\eta}))^{1/2}.$$

Since $\gamma(t) \geq 0$, a sufficient condition for inequality 3.15d to hold is that the nonnegative function $\alpha(t)$ satisfies

$$\alpha(t) < \frac{\pi^2 + \gamma(t)T^2}{K^2T^2\|\phi\|_2^2}.$$

If, in addition, η can be chosen equal to zero, then a sufficient condition for inequality 3.15c to hold is that

$$\alpha(t) < \frac{48(\gamma(t)T^2 + \pi^2)}{T^5\|\phi\|_2^2}.$$

One of the shortcomings of Theorems 3.7 and 3.8 is that the initial approximation $u_1(t)$ is chosen to be the zero function in all cases. It seems reasonable to assume that a more qualitative choice of $u_1(t)$, in some sense, would lead to more rapid convergence. If this is indeed the case, then how can $u_1(t)$ be chosen such that convergence of $\{u_k(t)\}$ is preserved? Is the convergence necessarily monotonic and what are the relative rates of convergence obtained by using different initial functions? Finally, is it possible to relax the conditions placed upon $B(t;u)$ and $\theta(t) = a(t)\omega'(t) + f(t,\omega)$ in Theorem 3.7 and still obtain some type of convergence? Consider

$$\varepsilon(t;u) = Lu - B(t;u)u(t) - a(t)\omega'(t) - f(t,\omega), \tag{3.16}$$

defined for every $u \in S \cap C^2[0, t]$.

Theorem 3.15: Suppose that there exists an $\eta \in [0, \pi^2/T^2)$ such that (3.8) holds and that there exists $u_1 \in S$ such that one of the following sets of conditions holds:

(i) $u_1(t) \leq 0$ on $[0, T]$; $\varepsilon(t;u_1) \leq 0$ on $[0, T]$; $B(t;u)$ is nondecreasing in u for all $u \in S$.

(ii) $u_1(t) \geq 0$ on $[0, T]$; $\varepsilon(t;u_1) \geq 0$ on $[0, T]$; $B(t;u)$ is nonincreasing in u for all $u \in S$.

Then the unique solution $v(t)$ of (3.5) which remains in S is the monotone limit of the sequence of successive approximations $\{u_k(t)\}$ defined by (3.14).

Proof: Existence of the unique $v \in S$ has been established in Theorem 3.2. Assume (i) and let $w_1 = u_2 - u_1$.

Then w_1 solves the boundary value problem

$$Lw_1 = B(t;u_1)w_1 - \varepsilon(t;u_1); \quad w_1(0) = w_1(T) = 0. \quad (3.17)$$

Therefore,

$$w_1(t) = \int_0^T G_1(t,s)\varepsilon(s;u_1(s))ds \le 0,$$

and it follows that $u_2(t) \le u_1(t) \le 0$. Let $w_k = u_{k+1} - u_k$, $k = 2,3,4,\cdots$, and assume $u_k \le u_{k-1} \le 0$, which implies that $w_{k-1} \le 0$. It can be shown that $w_k(t)$ solves the boundary value problem

$$Lw_k = B(t;u_k)w_k + (B(t;u_k) - B(t;u_{k-1}))u_k(t);$$
$$w_k(0) = w_k(T) = 0. \quad (3.18)$$

Therefore,

$$w_k(t) = -\int_0^T G_k(t,s)u_k(s)[B(s;u_k(s)) - B(s;u_{k-1}(s))]ds$$
$$\le 0. \quad (3.19)$$

Hence, $u_{k+1}(t) \le u_k(t) \le 0$, and $\{u_k(t)\}$ converges monotonically downward to $v(t)$. With assumption (ii), it can be shown analogously that $\{u_k(t)\}$ converges monotonically upward to $v(t)$. ||

<u>Theorem 3.16</u>: Suppose that there exists an $\eta \in [0, \pi^2/T^2)$ such that (3.8) holds and that $u_1(t) \in S$ such that one of the following sets of conditions holds:

(i) $0 \leq u_1(t) \leq u_3(t)$ on $[0, T]$; $\varepsilon(t;u_1) \geq 0$ on $[0, T]$; $B(t;u)$ is nondecreasing in u for all $u \in S$.

(ii) $0 \geq u_1(t) \geq u_3(t)$ on $[0, T]$; $\varepsilon(t;u_1) \leq 0$ on $[0, T]$; $B(t;u)$ is nonincreasing in u for all $u \in S$.

Then, respectively,

(i) $\{u_{2k+1}(t)\}$ and $\{u_{2k}(t)\}$ converge upward and downward respectively with $u_{2k+1}(t) \leq v(t) \leq u_{2k}(t)$.

(ii) $\{u_{2k}(t)\}$ and $\{u_{2k+1}(t)\}$ converge upward and downward respectively with $u_{2k}(t) \leq v(t) \leq u_{2k+1}(t)$.

Proof: Existence of the unique $v \in S$ was established in Theorem 3.2. Assume (i) and define the functions

$$w_k = u_{k+1} - u_k; \quad k = 1,2,3,\cdots, \tag{3.20a}$$

$$w_k^{(j)} = u_{k+1} - u_j; \quad k = 2,3,4,\cdots; \quad j = 1,2,3,\cdots; \quad j < k. \tag{3.20b}$$

Since, as in (3.17), $w_1(t)$ solves

$$Lw_1 = B(t;u_1)w_1 - \varepsilon(t;u_1); \quad w_1(0) = w_1(T) = 0,$$

it follows that

$$w_1(t) = \int_0^T G_1(t,s)\varepsilon(s;u_1(s))ds \geq 0$$

and, hence, $u_1(t) \leq u_2(t)$.

It can be shown that $w_k^{(j)}$ solves the boundary value problem

$$Lw_k^{(j)} = B(t;u_k)w_k^{(j)} + [B(t;u_k) - B(t;u_{j-1})]u_j(t),$$
$$w_k^{(j)}(0) = w_k^{(j)}(T) = 0; \quad k = 3,4,5,\cdots;$$
$$j = 2,3,4,\cdots; \quad j < k.$$

Therefore,

$$w_k^{(j)}(t) = -\int_0^T G_k(t,s)u_j(s)[B(s;u_k(s)) - B(s;u_{j-1}(s))]ds. \qquad (3.21)$$

Equations 3.21 and 3.19,

$$w_k(t) = -\int_0^T G_k(t,s)u_k(s)[B(s;u_k(s)) - B(s;u_{k-1}(s))]ds, \qquad (3.19)$$

are used frequently in the proof that follows. However, no specific mention is made when they are being used.

It has already been seen that $w_1 \geq 0$. Since $0 \leq u_1$, it follows that $0 \leq u_1 \leq u_2$. Therefore, $B(t;u_2) - B(t;u_1) \geq 0$ and, thus, $w_2 \leq 0$. Hence, $0 \leq u_1 \leq u_3 \leq u_2$, which implies that $B(t;u_3) - B(t;u_2) \leq 0$. Therefore, $w_3 \geq 0$ and, hence, $0 \leq u_1 \leq u_3 \leq u_4$. By assumption, $w_2^{(1)} \geq 0$ and, thus, $0 \leq u_1 \leq u_3$. This then implies that $B(t;u_3) - B(t;u_1) \geq 0$, which, in turn, implies that $w_3^{(2)} \leq 0$. $w_3^{(1)} \geq 0$ since $0 \leq u_1 \leq u_4$. The general induction proof can now be given. Let a step consist of the calculations of $w_k^{(k-1)}$ and w_k, necessarily in that order. Call this an even or an odd numbered step corresponding to k, even or odd. The above preliminary inequalities are used to verify steps 1 and 2 as follows:

<u>Step 1</u>: $w_4^{(3)} \geq 0$ since $B(t;u_4) - B(t;u_2) \leq 0$ and $u_3 \geq 0$. Thus, $0 \leq u_1 \leq u_3 \leq u_5$. $w_4 \leq 0$ since $B(t;u_5) - B(t;u_3) \geq 0$ and $u_4 \geq 0$. Thus, $0 \leq u_1 \leq u_3 \leq u_5 \leq u_4 \leq u_2$.

<u>Step 2</u>: $w_5^{(4)} \leq 0$ since $B(t;u_5) - B(t;u_3) \geq 0$ and $u_4 \geq 0$. Thus, $u_6 \leq u_4$. $w_5 \geq 0$ since $B(t;u_5) - B(t;u_4) \leq 0$ and $u_5 \geq 0$. Thus, $u_5 \leq u_6$ and, consequently, $0 \leq u_1 \leq u_3 \leq u_5 \leq u_6 \leq u_4 \leq u_2$.

Without loss of generality, suppose that k is even. Assume that $w_{2j+2}^{(2j+1)} \geq 0$, $w_{2j+2} \leq 0$, $w_{2j+1}^{(2j)} \leq 0$, and $w_{2j+1} \geq 0$ for all integers j such that $4 \leq 2j \leq k-2$. Then, consider steps $k+1$ and $k+2$.

<u>Step k+1</u>: $w_{k+1}^{(k)} \leq 0$ since $B(t;u_{k+1}) - B(t;u_{k-1}) \geq 0$ and $u_k \geq 0$. Thus, $u_{k+2} \leq u_k$. $w_{k+1} \geq 0$ since $B(t;u_{k+1}) - B(t;u_k) \leq 0$ and $u_{k+1} \geq 0$. Thus, $u_{k+1} \leq u_{k+2}$ and, consequently, $0 \leq u_1 \leq u_3 \leq \cdots \leq u_{k-1} \leq u_k \leq u_{k-2} \leq \cdots \leq u_4 \leq u_2$.

<u>Step k+2</u>: $w_{k+2}^{(k+1)} \geq 0$ since $B(t;u_{k+2}) - B(t;u_k) \leq 0$ and $u_{k+1} \geq 0$. Thus, $0 \leq u_1 \leq u_3 \leq \cdots \leq u_{k-1} \leq u_{k+1} \leq u_{k+3}$. $w_{k+2} \leq 0$ since $B(t;u_{k+2}) - B(t;u_{k+1}) \geq 0$ and $u_{k+2} \geq 0$. Thus, $0 \leq u_1 \leq u_3 \leq \cdots \leq u_{k-1} \leq u_{k+1} \leq u_{k+3} \leq u_{k+2} \leq u_k \leq u_{k-2} \leq \cdots \leq u_4 \leq u_2$.

Therefore, $\{u_{2k-1}\}$ and $\{u_{2k}\}$ converge respectively upward and downward monotonically.

Since $\varepsilon(t;u_1) \geq 0$, it follows that $u_1(t)$ is a lower solution for problem 3.5 and, thus, $u_1(t) \leq v(t)$. Thus, $B(t;u_1) \leq B(t;v)$, which implies that $G_1(t,s) \geq G(t,s;v)$. Therefore, $v(t) \leq u_2(t)$. Similarly, it follows that $u_{2k+1}(t) \leq v(t) \leq u_{2k}(t)$, and conclusion (i) follows.

Case (ii) is entirely analogous. ||

Note 3.17: If the hypothesis described in Note 3.9 is also assumed in Theorem 3.16, it follows that $\{u_{2k+1}(t)\}$ and $\{u_{2k}(t)\}$ converge to $v(t)$.

Note 3.18: The requirement in Theorems 3.7 and 3.8, namely the assumption that $a(t)\omega'(t) + f(t,\omega)$ does not change sign on $[0, T]$, is no longer assumed in Theorems 3.15 and 3.16. However, the choice of $u_1 \in S$ is much more restricted.

IV. OTHER RESULTS ON EXISTENCE AND UNIQUENESS

Although both the hypotheses and the conclusions of Theorem 3.2 are somewhat different from the usual theorems on existence and/or uniqueness of a solution to a boundary value problem, the method of proof is quite standard. This method of proving existence by using Ascoli's Theorem to extract a uniformly convergent subsequence of functions from a recursively generated sequence is further illustrated by the following theorem and its corollary. Proofs can be found in [10].

Theorem 4.1: Let $M > 0$ and $N > 0$ be given real numbers and let q be the maximum of $|h(t,y,y')|$ on the compact set $\{(t,y,y'): t \in [0, T], \ |y| \leq 2M, \ |y'| \leq 2N\}$. Then, if $\delta = \min\{(8M/q)^{1/2}, 2N/q\}$, (1.1) has a solution provided that $T < \delta$, $|A| \leq M$, $|B| \leq M$, $|\frac{B-A}{T}| \leq N$.

Corollary 4.2: Assume that there exist constants $h > 0$ and $k > 0$ such that $|h(t,y,y')| \leq h + k(|y|)^{1/2}$ on $[0, T] \times R^2$. Then (1.1) has a solution for all A and B.

The theory of sub(super)functions has played a significant role in the study of existence and/or uniqueness for (1.1). Although none of this theory of sub(super)functions will be given here as such, some of the results which have motivated the preceding work have been established in this fashion and are included in the present chapter. A fairly extensive development of sub(super)function theory, as well as a good bibliography, can be found in [10].

A standard assumption made when applying sub(super) function theory to (1.1) is that $h(t,y,y')$ be nondecreasing in y. The following theorem, which is a forerunner of Theorem 2.12, states that this is sufficient for both existence and uniqueness of a solution of (1.1) in case $h(t,y,y')$ is independent of y'. A proof can be found in [15].

<u>Theorem 4.3</u>: If $h(t,y,y')$ of (1.1) is independent of y' and is nondecreasing in y on $[0, T] \times R$, then (1.1) has a unique solution for all A and B.

Example 3.3 illustrates that (1.1) may not have a unique solution even though $h(t,y,y')$ is a relatively well-behaved function, In fact, it can be shown that if $h(t,y,y')$ is a quadratic polynomial in y, then there exists a T_o such that (1.1) will not have a solution for any $T \geq T_o$. An interesting result due to Schrader [16] is that such behavior cannot happen if $h(t,y,y')$ belongs to a certain class of polynomials in y.

<u>Theorem 4.4</u>: Assume that in (1.1) $h(t,y,y') = P(y)$ is a polynomial in y of odd degree. Then (1.1) has a unique solution for all choices of T, A, and B whenever

$P(y)$ is nonlinear.

The following result combines a local Lipschitz condition with the successive approximations defined by (3.14). This Lipschitz condition on $f(t,y)$ of (3.4) is perhaps more appropriately called a local Lipschitz condition on $f_y(t,y)$ rather than on $f(t,y)$ itself. The term local Lipschitz has the standard meaning, namely, that the Lipschitz assumption is only for functions y contained in some specified class of functions. The particular Lipschitz assumption placed upon $f(t,y)$ can be thought of intuitively as a bounded concavity assumption. The condition is

$$|B(t;u) - B(t;v)| \leq L|u-v| \tag{4.1}$$

for all $u,v \in S$ and fixed $t \in [0, T]$.

<u>Theorem 4.5</u>: Recall (3.1a-b) and suppose that (3.11) is disconjugate on $[0, T]$ for all $u \in S$. Assume also that (4.1) holds and that

$$CK^2L\sqrt{T} < 1, \tag{4.2}$$

where

$$C = \|g\|_2 \max_{0\leq t\leq T} (\exp(\tfrac{1}{2} \textstyle\int_0^t a(s)ds)). \tag{4.3}$$

Then (3.5) has a unique solution $v \in S$, and the successive approximations defined by (3.14) converge to v for any $u_1 \in S$. Furthermore, an estimate on the rate of convergence is given by

$$\|u_{j+2} - u_{j+1}\|_\infty \leq (CK^2L\sqrt{T})^j \|u_2 - u_1\|_\infty . \qquad (4.4)$$

<u>Proof</u>: The map Γ defined by (3.10) can be written as $\Gamma u_k = u_{k+1}$ for any indexed function $u_k \in S$, where u_{k+1} is defined by (3.14). The disconjugacy assumption for (3.11) guarantees that Γ is well-defined and, furthermore, as in Theorem 3.2, Γ is uniformly continuous on S. With $w_j = u_{j+1} - u_j$, $j=2,3,4,\cdots$, and $K = \max_{0\leq t\leq T} \|G(t,\cdot;-\eta\|_2$ we have

$$\begin{aligned} |w_j| &\leq KC\int_0^T G_j(t,s)\,|B(s;u_j(s)) - B(s;u_{j-1}(s))|\,ds \\ &\leq CK^2L\sqrt{T}\ \|w_{j-1}\|_\infty ; \quad j = 2,3,4,\cdots. \end{aligned}$$

Thus, Γ is a contraction on S and, since S is a complete metric subspace of $C[0, T]$, Γ has a unique fixed point $v \in S$. The estimate (4.4) is obtained by repeated applications of the above argument. ||

One practical method of showing that (3.11) is disconjugate on $[0, T]$ for all $u \in S$ is to establish (3.9). An observation to be made from (4.2) is that for T sufficiently small, the boundary value problem (3.5) can be solved by the method of successive approximations defined by (3.14), provided that K is assumed to exist and be finite and that $f_{yy}(t,y)$ exists and is continuous on $[0, T] \times R$. This observation becomes very evident in the particular application of Theorem 4.5 with the assumptions $a(t) \equiv 0$ on $[0, T]$ and $B(t;u) \geq -\eta = 0$ for all $u \in S$.

Corollary 4.6: If in (3.4) $a(t) \equiv 0$ on $[0, T]$ and in (3.9) $\eta = 0$, then (4.2) becomes

$$LT^{7/2} \|f(t,\omega)\|_2 < 48. \tag{4.5}$$

Proof: Observe that $g(t) = f(t,\omega)$ and that $K = \frac{T^{3/2}\sqrt{3}}{12}$. ||

As an illustration of Corollary 4.6, recall Example 3.14. If $f(t,y) = \alpha y^2 + \beta y + \gamma$, then the Lipschitz condition (4.1) is satisfied with $L = |\alpha|$. Furthermore, if $A = B = 0$ and $\eta = 0$, the hypotheses of Corollary 4.6 are satisfied if

$$\beta \geq \frac{|\alpha\gamma| T^2\sqrt{3}}{12}$$

and

$$|\alpha\gamma| T^{7/2} < 48.$$

Thus, any initial approximation $u_1(t)$ can be chosen from the set

$$S = \{y \in C[0, T]: y(0) = y(T) = 0; \quad |y| \leq \frac{|\gamma| T^2\sqrt{3}}{12}\}$$

and convergence of the successive approximations (3.14) is assured.

V. CONCLUDING REMARKS

During the preparation of this thesis, several questions have arisen. Included in the following list are questions which are believed to be presently unanswered, as well as questions of interest concerning the methods of proof used in some of the theorems.

(1) As has already been stated in Note 3.5, Theorem 3.2 would be greatly improved if it could be shown that (3.9) were to guarantee uniqueness of a solution to (3.5) within the set S. A careful analysis of Example 3.3, along with a computer print-out of the second solution to the given problem, shows that there is an optimal η which is close to $\pi^2/4$ such that (3.9) is satisfied on a set S which is as large as possible. However, S is not quite large enough to contain the second solution. Although this is only one example, nevertheless, it seems to indicate that if the conjecture is true, the result may be best possible.

(2) With reference to Theorem 3.7, it is conjectured that if $a(t)\omega'(t) + f(t,\omega)$ changes sign once at $t_o \in (0, T)$, then each successive approximation will also change sign once on $[0, T]$, but not necessarily at t_o. If this were true, then the boundary value problem (3.5) could be reduced to two boundary value problems, the solution of each being the monotone limit of the successive approximations (3.14) defined on the respective subintervals. One of the

major difficulties involved is that the two smaller boundary value problems will not, in general, have zero boundary conditions at 0, t_o, and T. In fact, the boundary conditions will generally not even be known at t_o. This forces the quasi-linearizations of the smaller problems to contain the straight line joining the boundary data as a parameter. If this process could be carried out successfully when $a(t)\omega'(t) + f(t,\omega)$ has one zero on $[0, T]$, then it could be generalized to the n-zero case.

(3) A computer print-out for inequality (2.38a) on the interval $[.2, 1.57]$ at increments of .01 shows that not only is

$$f(x) = \frac{8\pi^2 x^3}{(\pi^2-4x^2)^2} + \tan x - x \sec^2 x$$

a positive function on this interval, but also that $f(x)$ increases steadily from approximately .02 to over 16,844.

(4) A weaker version of Theorem 2.6 can be proved by elementary methods without using the theory of sub(super) functions. This elementary proof is given here, and the question is asked whether or not Theorem 2.6, as given in Chapter 2, can be proved in the same manner.

Define the operators $L_j\colon C^2[0, T] \to C[0, T]$ by

$$L_j y = y'' - a_j(t)y' - b_j(t)y; \quad j = 1,2, \tag{5.1}$$

where $a_j, b_j \in C[0, T]$, $j = 1,2$.

Lemma 5.1: Let $u_j(t)$ solve $L_j y = 0$, $j = 1,2$, with the initial conditions (2.6), $j = 1$. [$v_j(t)$ solve $L_j y = 0$, $j = 1,2$, with the initial conditions (2.6), $j = 2$]. If $a_1(t) \leq a_2(t)$ and $0 \leq b_1(t) \leq b_2(t)$ on $[0, T]$, then on $[0, T]$,

(i) $0 \leq u_1(t) \leq u_2(t)$ [$1 \leq v_1(t) \leq v_2(t)$], and

(ii) $0 \leq u_1'(t) \leq u_2'(t)$ [$0 \leq v_1'(t) \leq v_2'(t)$].

Proof: By integrating the self-adjoint form of $L_j y = 0$ from 0 to t, the assumption to the contrary of (i) is seen to be impossible. With $w = u_2'u_1 - u_2u_1'$, it follows that

$$(w \cdot \exp(-\int_0^t a_2(s)ds))' = (u_2u_1(b_2 - b_1) + u_2u_1'(a_2 - a_1))\exp(-\int_0^t a_2(s)ds), \tag{5.2}$$

and similarly for the function $w = v_2'v_1 - v_2v_1'$. Therefore, $w(t) \geq 0$ in each case, and (ii) follows. ||

Lemma 5.2: With the assumptions of Lemma 5.1,

$$u_1'u_2 \leq u_2'u_1 \quad [v_1'v_2 \leq v_2'v_1]$$

on $[0, T]$.

Proof: The result follows directly from (5.2). ||

Lemma 5.3: With the assumptions of Lemma 5.1, let

$$\theta_j(t) = u_j(t)v_j(T) - v_j(t)u_j(T), \qquad j = 1,2.$$

Then, $\theta_j(t)$, $j = 1,2$, are nondecreasing nonpositive functions on $[0, T]$.

Proof: By integrating the self-adjoint differential equation which $\theta_j(t)$ solves, namely,

$$(\theta_j' \exp(\textstyle\int_t^T a_j(s)ds))' = b_j \theta_j \exp(\textstyle\int_t^T a_j(s)ds),$$

from t to T, the result follows by an indirect argument. ||

Lemma 5.4: Let $u_j(t)$ and $v_j(t)$ be as defined in Lemma 5.1. If $a_2(t) \leq a_1(t)$ and $0 \leq b_1(t) \leq b_2(t)$ on $[0, T]$, then $\theta_1\theta_2' \leq \theta_2\theta_1'$ on $[0, T]$.

Proof: Let $u = \theta_1\theta_2' - \theta_2\theta_1'$ and integrate the equation

$$u \exp(\textstyle\int_t^T a_2(s)ds)' = (\theta_1\theta_2(b_2-b_1) + \theta_2\theta_1'(a_2-a_1))\exp(\textstyle\int_t^T a_2(s)ds)$$

from t to T to obtain the result. ||

Theorem 5.5: Let $G(t,s;b_j)$ denote the Green's function for the problem (2.13), i.e., for $Ly = b_j(t)y$, $j=1,2$, where L is defined by (2.1). If $0 \leq b_1(t) \leq b_2(t)$ on $[0, T]$, then (2.14) holds on $[0, T] \times [0, T]$, namely,

$$0 \leq G(t,s;b_2) \leq G(t,s;b_1).$$

<u>Proof</u>: Let $\alpha(t) = G(t,t;b_1) - G(t,t;b_2)$ and $r(t) = \exp(\int_0^t a_1(x)dx)$. Then

$$r\alpha = u_1(v_1 - k_1u_1) - u_2(v_2 - k_2u_2),$$

where $k_j = v_j(T)/u_j(T)$, $j = 1,2$ and, moreover,

$$(r\alpha)' = u_1'v_1 + u_1v_1' - u_2'v_2 - u_2v_2' - 2k_1u_1u_1' + 2k_2u_2u_2'.$$

Since $a_1(t) \equiv a_2(t)$ on $[0, T]$, it follows that

$$(r\alpha)' = 2u_1(v_1' - k_1u_1') - 2u_2(v_2' - k_2u_2').$$

If $(r\alpha)'(t_o) = 0$ for some $t_o \in (0, T)$, then at $t = t_o$, it can be shown that

$$(r\alpha)(t_o) = \frac{u_2(v_2' - k_2u_2')}{v_1' - k_1u_1'}(v_1 - k_1u_1) - u_2(v_2 - k_2u_2). \qquad (5.3)$$

It can be shown by an elementary argument that the denominator of (5.3) is not zero. Therefore, at $t = t_o$, it follows that

$$\begin{aligned}(r\alpha)(t_o) &= u_2(v_2' - k_2u_2')\left(\frac{v_1 - k_1u_1}{v_1' - k_1u_1'} - \frac{v_2 - k_2u_2}{v_2' - k_2u_2'}\right)\\ &= u_2\left(\frac{-\theta_2'}{u_2(T)}\right)\left(\frac{\theta_1}{\theta_1'} - \frac{\theta_2}{\theta_2'}\right)\\ &\geq 0.\end{aligned}$$

Thus, since $\alpha(0) = \alpha(T) = 0$, inequality (2.14) is established on the diagonal of $[0, T] \times [0, T]$. For any (t,s) $[0, T] \times [0, T]$ with $s \leq t$, it follows, since $\alpha(t) \geq 0$, that

$$\frac{u_1(T)}{u_2(T)} \cdot \frac{u_2(t)}{u_1(t)} \leq \frac{\theta_1(t)}{\theta_2(t)} .$$

Therefore, by Lemmas 5.1 and 5.2, since $s \leq t$,

$$\frac{u_1(T)}{u_2(T)} \frac{u_2(s)}{u_1(s)} \leq \frac{u_1(T)}{u_2(T)} \frac{u_2(t)}{u_1(t)} \leq \frac{\theta_1(t)}{\theta_2(t)} .$$

Finally, Lemma 5.3 implies that

$$e^{-\int_o^s a_2(x)dx} \cdot \frac{-u_2(s)}{u_2(T)} \quad \theta_2(t) \leq e^{-\int_o^s a_1(x)dx} \cdot \frac{-u_1(s)}{u_1(T)} \theta_1(t) ,$$

which is precisely inequality (2.14). The proof is similar for $t \leq s$. ||

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